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Some Polynomial Projections with Finite Carrier

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1. INTRODUCTION AND PRELIMINARIES

Let $C[-1, 1]$ denote the space of all continuous real-valued functions defined on $[-1, 1]$, with the norm $\|f\|_\infty = \max_{-1 \leqslant x \leqslant 1} |f(x)|$. Denote by Π_n the subspace of $C[-1, 1]$ consisting of all polynomials of degree at most n. A projection of $C[-1, 1]$ onto Π_n is a bounded linear map P: $C[-1, 1] \to \Pi_n$ such that $Pu = u$ for all $u \in \Pi_n$. Let $\mathscr{P}_n$ be the family of all such projections. Every $P \in \mathscr{P}_n$ has the property

$$\|f - Pf\|_\infty \leqslant (1 + \|P\|) E_n(f), \tag{1.1}$$

where $E_n(f)$ is the error of the best approximation of f by elements of Π_n. A projection P with small $\|P\|$ will, therefore, provide a good approximation to f. It is known [8] that there exists $P^* \in \mathscr{P}_n$ such that $\|P^*\| \leqslant \|P\|$ for all $P \in \mathscr{P}_n$. Such a P^* is called a minimal projection from the class $\mathscr{P}_n$. Minimal projections for $n \geqslant 2$ remain unknown.

An important example of projection is the Fourier–Chebyshev operator S_n given by

$$S_n f = \sum_{k=0}^{n}{}' a_k[f]\, T_k, \tag{1.2}$$

where

$$a_k[f] = \frac{2}{\pi} \int_0^\pi f(\cos t) \cos kt\, dt, \qquad (k = 0, 1, \ldots).$$
$$T_k(x) = \cos(k \arccos x)$$

The symbol $\sum'$ denotes the sum with the first term halved. It is well-known (see, e.g., [9]) that

$$\|S_n\| = \frac{1}{2n+1} + \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k} \tan \frac{k\pi}{2n+1}. \tag{1.3}$$

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The norm of S_n increases with n rather slowly: $\|S_n\| \leqslant 3.14$ for $n \leqslant 100$ (cf. Table III; values of $\|S_n\|$ were taken from [9]). In view of (1.1), the polynomial $S_n f$ is a near-best polynomial approximation to f by elements of Π_n. It was proved by Cheney and Rivlin [5] that S_n is not a minimal projection from $\mathscr{P}_n$.

Let $\Omega_m = \{x_0, x_1, \dots, x_m\}$ be a prescribed set of m $(m \geqslant n)$ pair-wise different points of $[-1, 1]$. Denote by $\mathscr{D}_n(\Omega_m)$ the subclass of $\mathscr{P}_n$ containing projections carried by Ω_m. Any projection belonging to this subclass has the form [3]

$$Pf = \sum_{k=0}^{m} f(x_k)\, w_k \qquad (f \in C[-1, 1]) \tag{1.4}$$

for appropriate $w_0, w_1, \dots, w_m \in \Pi_n$. The function

$$\Lambda_p(x) = \sum_{k=0}^{m} |w_k(x)| \tag{1.5}$$

is called the Lebesgue function of the operator P. It is well-known that

$$\|P\| = \|\Lambda_p\|_\infty . \tag{1.6}$$

In this paper we discuss two special choices of carriers, namely

$$\Omega_m^t = \{t_{m+1,1}, t_{m+1,2}, \dots, t_{m+1,m+1}\}, \tag{1.7}$$

where

$$t_{m+1,j} = \cos \frac{2j-1}{2m+2}\pi \qquad (j = 1, 2, \dots, m+1)$$

(zeros of the Chebyshev polynomial T_{m+1}), and

$$\Omega_m^u = \{u_{m-1,0}, u_{m-1,1}, \dots, u_{m-1,m}\}, \tag{1.8}$$

where

$$u_{m-1,j} = \cos \frac{j\pi}{m} \qquad (j = 0, 1, \dots, m)$$

(extreme points of the Chebyshev polynomial T_m).

Class $\mathscr{D}_n(\Omega_m^t)$ contains the operator $I_n^{(m)}$ defined by

$$I_n^{(m)} f = \sum_{k=0}^{n}{}' \alpha_k^{(m)}[f]\, T_k \qquad (m \geqslant n), \tag{1.9}$$

where

$$\alpha_k^{(m)}[f] = \frac{2}{m+1} \sum_{j=1}^{m+1} f(t_{m+1,j})\, T_k(t_{m+1,j}) \qquad (k = 0, 1, \ldots, m).$$

The operator $J_n^{(m)}$ such that

$$J_n^{(m)} f = \sum_{k=0}^{n}{}^{*} \beta_k^{(m)}[f]\, T_k \qquad (m \geqslant n), \tag{1.10}$$

where

$$\beta_k^{(m)}[f] = \frac{2}{m} \sum_{j=0}^{m}{}'' f(u_{m-1,j})\, T_j(u_{m-1,k}) \qquad (k = 0, 1, \ldots, m),$$

belongs to the class $\mathscr{D}_n(\Omega_m^u)$. Here the symbol $\sum''$ denotes the sum with the first and the last terms halved. The symbol $\sum^*$ is equivalent to the symbol $\sum''$ if $m = n$, or to the symbol $\sum'$ if $m > n$.

For $m > n$ the polynomials $I_n^{(m)} f$ and $J_n^{(m)} f$ are the least-squares approximations to f on the point sets Ω_m^t and Ω_m^u, respectively. For $m = n$ formulas (1.9) and (1.10) define operators

$$I_n = I_n^{(n)}, \qquad J_n = {}_n^{(n)} \tag{1.11}$$

such that polynomials $I_n f$ and $J_n f$ interpolate to f at the points $t_{n+1,j}$ $(j = 1, 2, \ldots, n+1)$ and $u_{n-1,k}$ $(k = 0, 1, \ldots, n)$, respectively. Norms of I_n and J_n are given explicitly by (see [6, 7, 9])

$$\|I_n\| = \frac{1}{n+1} \sum_{k=1}^{n+1} \csc \frac{(2k-1)\pi}{2n+2} = \frac{1}{n+1} \sum_{k=1}^{n+1} \tan \frac{(2k-1)\pi}{4n+4}, \tag{1.12}$$

$$\begin{aligned} \|J_n\| &= \|I_{n-1}\| && (n = 1, 3, \ldots), \\ &= \|I_{n-1}\| - \varepsilon_n && (n = 2, 4, \ldots), \end{aligned}$$

where $0 < \varepsilon_n < (1/n)\tan(\pi/4n)$.

Some values of $\|I_n^{(m)}\|$ and $\|J_n^{(m)}\|$ are tabulated in Tables I and II (see also Table 1 in [1]). We see that in practice the polynomials $I_n^{(m)} f$ and $J_n^{(m)} f$ can serve as near-best minimax approximations to f. However, more careful examination leads to the discovery that

$$\begin{aligned} \|I_n^{(m)}\| &\geqslant \|I_n^{(n+[n/2])}\| && (n = 1, 2, \ldots), \\ \|J_n^{(m)}\| &\geqslant \|J_n^{(2n+1)}\| && (n = 3, 4, \ldots) \end{aligned}$$

for all $m \geqslant n$.

TABLE I

Values of $\|I_n^{(m)}\|$

	n							
$m-n$	1	2	3	4	5	10	15	20
0	1.414	1.667	1.848	1.989	2.104	2.489	2.728	2.901
1	1.488	1.590	1.722	1.876	1.951	2.303	2.519	2.684
2	1.424	1.717	1.748	1.808	1.887	2.219	2.433	2.591
3	1.431	1.626	1.869	1.859	1.919	2.229	2.391	2.536
4	1.449	1.638	1.772	1.982	1.951	2.169	2.372	2.511
5	1.432	1.643	1.756	1.880	2.072	2.121	2.359	2.483
6	1.434	1.633	1.780	1.856	1.968	2.175	2.329	2.485
7	1.442	1.660	1.784	1.872	1.950	2.195	2.266	2.466
8	1.434	1.637	1.768	1.887	1.933	2.212	2.303	2.444
9	1.435	1.641	1.773	1.890	1.951	2.245	2.332	2.410
10	1.439	1.642	1.800	1.878	1.972	2.363	2.353	2.367
11	1.435	1.639	1.776	1.867	1.975	2.254	2.346	2.424
12	1.435	1.650	1.772	1.877	1.958	2.230	2.370	2.445
13	1.438	1.640	1.779	1.905	1.948	2.218	2.387	2.465
14	1.435	1.642	1.780	1.879	1.954	2.209	2.421	2.464
15	1.435	1.642	1.774	1.873	1.961	2.187	2.538	2.460
20	1.436	1.642	1.779	1.875	1.958	2.247	2.376	2.664
30	1.436	1.642	1.778	1.880	1.957	2.229	2.412	2.450

Let us denote

$$L_n = I_n^{(n+[n/2])}, \tag{1.13}$$

$$M_n = J_n^{(2n+1)}. \tag{1.14}$$

Our objective here is to study these operators. The obtained results—bounds for the norms $\|L_n\|$ and $\|M_n\|$ (see Sections 2 and 3) and their numerical values for $n = 1, 2, \ldots, 20, 50, 100$ (see Table III)—show that these projections may be of some interest.

It follows from the results of Cheney [3] that any projection P from the set $\mathscr{D}_n(\Omega^t_{n+[n/2]})$ can be represented in the form

$$P = L_n + Q,$$

where Q is such that

$$Qf = \sum_{k=1}^{[n/2]} \alpha_{n+k}^{(n+[n/2])}[f]\, u_k$$

TABLE II

Values of $\|J_n^{(m)}\|$

	n							
$m-n$	1	2	3	4	5	10	15	20
0	1.000	1.250	1.667	1.799	1.989	2.421	2.687	2.869
1	1.500	1.667	1.830	1.989	2.094	2.489	2.726	2.901
2	1.333	1.707	1.847	1.911	2.024	2.191	2.407	2.573
3	1.457	1.494	1.833	1.942	2.023	2.260	2.366	2.471
4	1.447	1.667	1.601	1.924	2.024	2.235	2.415	2.487
5	1.411	1.656	1.799	1.681	1.994	2.286	2.437	2.525
6	1.443	1.648	1.801	1.894	1.745	2.267	2.443	2.544
7	1.440	1.654	1.784	1.903	1.969	2.288	2.450	2.549
8	1.425	1.606	1.779	1.893	1.985	2.274	2.416	2.561
9	1.439	1.650	1.790	1.880	1.982	2.258	2.410	2.564
10	1.438	1.646	1.789	1.875	1.981	2.214	2.445	2.551
11	1.430	1.645	1.735	1.886	1.955	1.984	2.432	2.519
12	1.438	1.647	1.785	1.891	1.951	2.199	2.429	2.557
13	1.437	1.626	1.785	1.888	1.971	2.232	2.416	2.558
14	1.432	1.646	1.780	1.831	1.972	2.242	2.393	2.544
15	1.437	1.644	1.779	1.885	1.973	2.248	2.343	2.550
16	1.437	1.644	1.783	1.887	1.967	2.240	2.153	2.538
17	1.433	1.645	1.782	1.884	1.908	2.245	2.333	2.529
18	1.437	1.633	1.759	1.879	1.964	2.232	2.373	2.514
19	1.437	1.644	1.781	1.878	1.968	2.240	2.390	2.487
20	1.434	1.643	1.782	1.883	1.968	2.228	2.397	2.435
21	1.437	1.643	1.779	1.885	1.967	2.196	2.399	2.276
22	1.437	1.644	1.779	1.883	1.959	2.194	2.405	2.427
23	1.434	1.636	1.781	1.859	1.958	2.222	2.403	2.471
24	1.436	1.644	1.781	1.882	1.965	2.231	2.393	2.492
25	1.436	1.643	1.768	1.883	1.966	2.227	2.404	2.503
30	1.436	1.643	1.780	1.883	1.965	2.229	2.376	2.518

for appropriate $u_1, u_2, ..., u_{[n/2]} \in \Pi_n$. Similarly, if R belongs to the class $\mathscr{D}_n(\Omega^u_{2n+1})$ then we have

$$R = M_n + S,$$

where S is such that

$$Sf = \sum_{k=1}^{n+1} \beta_{n+k}^{(2n+1)}[f]\, v_k$$

for some $v_1, v_2, ..., v_{n+1} \in \Pi_n$.

TABLE III

n	$\|S_n\|$	$\|I_n\|$	$\|J_n\|$	$\|L_n\|$	$\|M_n\|$
1	1.436	1.414	1.000	1.414	1.333
2	1.642	1.667	1.250	1.590	1.494
3	1.778	1.848	1.667	1.722	1.601
4	1.880	1.989	1.799	1.808	1.681
5	1.961	2.104	1.989	1.887	1.745
6	2.029	2.202	2.082	1.944	1.798
7	2.087	2.287	2.202	2.000	1.843
8	2.138	2.362	2.275	2.043	1.893
9	2.183	2.429	2.362	2.086	1.939
10	2.223	2.489	2.421	2.121	1.984
11	2.260	2.545	2.489	2.156	2.022
12	2.294	2.596	2.539	2.185	2.059
13	2.325	2.643	2.596	2.215	2.092
14	2.354	2.687	2.639	2.240	2.124
15	2.381	2.728	2.687	2.266	2.153
16	2.406	2.766	2.725	2.287	2.181
17	2.430	2.803	2.766	2.310	2.206
18	2.453	2.837	2.800	2.329	2.231
19	2.474	2.870	2.837	2.350	2.254
20	2.494	2.901	2.868	2.367	2.276
50	2.860	3.466	3.453	2.699	2.677
100	3.139	3.901	3.894	2.952	2.985

Now, the problem is to select the polynomials $u_1, u_2, \ldots, u_{[n/2]}$ ($v_1, v_2, \ldots, v_{n+1}$, respectively) in such a way that the norm of P (R, respectively) is as small as possible. The techniques discussed in [2] and [4] appear to be applicable. However, we do not discuss this problem here.

2. The Operator L_n

The operator L_n defined by (1.13) has, in view of (1.9), the form

$$(L_n f)(x) = \frac{2}{N+1} \sum_{k=0}^{n}{}' \left(\sum_{j=1}^{N+1} f(t_{N+1,j})\, T_k(t_{N+1,j}) \right) T_k(x), \qquad (2.1)$$

where $N = n + [n/2]$, $t_{N+1,j} = \cos((2j-1)/(2N+2))\pi$ $(j = 1, 2, \ldots, N+1)$. The Lebesgue function of this operator can be written in a variety of ways, for example,

$$\Lambda_{L_n}(x) = \frac{2}{N+1} \sum_{j=1}^{N+1} \left| \sum_{k=0}^{n}{}' T_k(t_{N+1,j})\, T_k(x) \right|$$
$$= \frac{2}{N+1} \sum_{j=1}^{N+1} \left| \sum_{k=0}^{n}{}' \cos k\theta_j \cos kt \right|$$
$$= \frac{1}{N+1} \sum_{j=1}^{N+1} |D_n(t-\theta_j) + D_n(t+\theta_j)|. \tag{2.2}$$

Here $\theta_j = ((2j-1)/(2N+2))\,\pi$ $(j = 1, 2,..., N+1)$, $\cos t = x$, and

$$D_n(u) = \sum_{k=0}^{n}{}' \cos ku = \frac{\sin(n+1/2)\,u}{2\sin u/2}. \tag{2.3}$$

We prove the following

THEOREM 2.1. *Operator L_n satisfies the inequalities*

$$\gamma_n \leqslant \|L_n\| \leqslant \|S_n\| \cdot \|I_N\|, \tag{2.4}$$

where $N = n + [n/2]$,

$$\gamma_n = \frac{1}{2\sqrt{3}} \left\{ 3\,\|I_{3N+2}\| - \|I_N\| - \frac{2}{n+1} \sum_{k=1}^{[(n+1+(-1)^n)/2]} \tan \frac{6k-2(-1)^n-3}{12(N+1)}\,\pi \right\}, \tag{2.5}$$

and I_N, I_{3N+2} are the interpolation operators defined as in (1.11), *S_n is the Fourier–Chebyshev operator given by* (1.2).

Proof. First we prove the left inequality of (2.4). Obviously

$$\|L_n\| = \|\Lambda_{L_n}\|_\infty \geqslant \Lambda_{L_n}(1).$$

We show that

$$\Lambda_{L_n}(1) = \gamma_n, \tag{2.6}$$

γ_n being defined in (2.5). From (2.2) one obtains

$$\Lambda_{L_n}(1) = \frac{1}{N+1} \sum_{k=1}^{N+1} |D_n(\theta_k)| = \frac{1}{N+1} \sum_{k=1}^{N+1} \left| \sin\left(n+\frac{1}{2}\right)\theta_k \right| \Big/ \sin\frac{1}{2}\theta_k. \tag{2.7}$$

If n is odd, $n = 2p + 1$, then we have

$$\Lambda_{L_{2p+1}}(1) = \frac{1}{3p+2} \sum_{k=1}^{3p+2} \left| \sin\left(\frac{2k-1}{3}\pi + \frac{1}{6}\theta_k\right) \right| \Big/ \sin\frac{1}{2}\theta_k.$$

Using first the identity

$$\begin{aligned} \sin\left(\alpha + \frac{2k-1}{3}\pi\right) &= \sin(\alpha + \pi/3) \qquad (k = 3l-2), \\ &= -\sin\alpha \qquad (k = 3l-1), \\ &= \sin(\alpha - \pi/3) \qquad (k = 3l), \end{aligned} \tag{2.8}$$

and then the identity

$$\frac{\cos\theta}{\cos 3\theta} = \frac{1}{2\sqrt{3}} [\tan(\pi/3 - \theta) + \tan(\pi/3 + \theta)], \tag{2.9}$$

we obtain

$$\begin{aligned} \Lambda_{L_{2p+1}}(1) &= \frac{1}{2\sqrt{3}(3p+2)} \left\{ \sum_{k=1}^{p+1} \left[\tan\frac{1}{6}(\pi + \theta_{3k-2}) + \tan\frac{1}{2}\left(\pi - \frac{1}{3}\theta_{3k-2}\right) \right.\right. \\ &\quad \left. + \tan\frac{1}{6}(\pi - \theta_{3k-1}) + \tan\frac{1}{6}(\pi + \theta_{3k-1}) \right] \\ &\quad \left. + \sum_{k=1}^{p} \left[\tan\frac{1}{2}\left(\pi - \frac{1}{3}\theta_{3k}\right) - \tan\frac{1}{6}(\pi - \theta_{3k}) \right\} \\ &= \frac{1}{2\sqrt{3}(3p+2)} \left\{ \sum_{k=1}^{9p+6} \tan\frac{2k-1}{4(9p+6)}\pi - \sum_{k=1}^{3p+2} \tan\frac{2k-1}{4(3p+2)}\pi \right. \\ &\quad \left. - 2\sum_{k=1}^{p} \tan\frac{6k-1}{12(3p+2)}\pi \right\} \\ &= \frac{1}{2\sqrt{3}} \left\{ 3\,\|I_{9p+5}\| - \|I_{3p+1}\| - \frac{2}{3p+2} \sum_{k=1}^{p} \tan\frac{6k-1}{12(3p+2)}\pi \right\} \\ &= \gamma_{2p+1} \end{aligned}$$

(cf. (1.12) and (2.5)).

If n is even, $n = 2p$, from (2.7) follows the equation

$$\Lambda_{L_{2p}}(1) = \frac{1}{3p+1} \sum_{k=1}^{3p+1} \left| \sin\left(\frac{2k-1}{3}\pi - \frac{1}{6}\theta_k\right) \right| \Big/ \sin\frac{1}{2}\theta_k,$$

which, together with identities (2.8) and (2.9), implies

$$\Lambda_{L_{2p}}(1) = \frac{1}{2\sqrt{3}(3p+1)} \left\{ \sum_{k=1}^{9p+3} \tan \frac{2k-1}{4(9p+3)}\pi - \sum_{k=1}^{3p+1} \tan \frac{2k-1}{4(3p+1)}\pi \right.$$
$$\left. - 2\sum_{k=1}^{p+1} \tan \frac{6k-5}{12(3p+1)}\pi \right\}$$
$$= \frac{1}{2\sqrt{3}} \left\{ 3\|I_{9p+2}\| - \|I_{3p}\| - \frac{2}{3p+1} \sum_{k=1}^{p+1} \tan \frac{6k-5}{12(3p+1)}\pi \right\}$$
$$= \gamma_{2p}.$$

Formula (2.6) is proved to be true.

The proof of the right inequality of (2.4) is very simple. It can be easily seen that

$$I_n^{(m)} = S_n I_m$$

(cf. (1.9), (1.11) and (1.2)). In particular,

$$L_n = I_n^{(N)} = S_n I_N.$$

Thus $\|L_n\| \leqslant \|S_n\| \cdot \|I_N\|$. ∎

From the results of numerical calculations, reported in Table III, one can conjecture that

$$\|L_n\| = \gamma_n,$$

γ_n being defined by (2.5).

3. The Operator M_n

The operator M_n defined by (1.14) has the values given by

$$(M_n f)(x) = \frac{2}{2n+1} \sum_{j=0}^{n}{}' \left(\sum_{k=0}^{2n+1}{}'' f(u_{2n,k})\, T_j(u_{2n,k}) \right) T_j(x) \tag{3.1}$$

(cf. (1.10)), where $u_{2n,k} = \cos((k\pi/(2n+1))$ $(k = 0, 1,..., 2n+1)$.

Some of the alternative forms of the Lebesgue function of M_n are

$$\Lambda_{M_n}(x) = \frac{2}{2n+1} \sum_{j=0}^{2n+1}{}'' \left| \sum_{k=0}^{n}{}' T_k(u_{2n,j})\, T_k(x) \right|$$
$$= \frac{2}{2n+1} \sum_{j=0}^{2n+1}{}'' \left| \sum_{k=0}^{n}{}' \cos k\theta_j \cos kt \right|$$

$$
\begin{aligned}
&= \frac{1}{2n+1} \sum_{j=0}^{2n+1}{}'' |D_n(t+\theta_j) + D_n(t-\theta_j)| \\
&= \frac{2}{2n+1} \left\{ \left| \sin\left(n+\frac{1}{2}\right)t \right| \sin\frac{t}{2} \sum_{j=0}^{n}{}' \frac{\cos\theta_j}{|\cos t - \cos\theta_{2j}|} \right. \\
&\quad \left. + \left| \cos\left(n+\frac{1}{2}\right)t \right| \cos\frac{t}{2} \sum_{j=0}^{n}{}' \frac{\cos\theta_j}{|\cos t + \cos\theta_{2j}|} \right\} \\
&= \frac{1}{4(2n+1)} \left\{ \left| \sin\left(n+\frac{1}{2}\right)t \right| \sum_{k=0}^{n}{}' \left| \tan\frac{1}{4}(t-\theta_{2k}) \right. \right. \\
&\quad + \cot\frac{1}{4}(t-\theta_{2k}) \\
&\quad \left. + \tan\frac{1}{4}(t+\theta_{2k}) + \cot\frac{1}{4}(t+\theta_{2k}) \right| + \left| \cos\left(n+\frac{1}{2}\right)t \right| \\
&\quad \times \sum_{k=0}^{n}{}^{\wedge} \left| \tan\frac{1}{4}(\theta_{2k+1}-t) \right. \\
&\quad \left. \left. + \cot\frac{1}{4}(\theta_{2k+1}-t) + \tan\frac{1}{4}(\theta_{2k+1}+t) + \cot\frac{1}{4}(\theta_{2k+1}+t) \right| \right\}.
\end{aligned}
\tag{3.2}
$$

Here $\theta_j = j\pi/(2n+1)$, $\cos t = x$, D_n is defined by (2.3), and the symbol $\sum^{\wedge}$ denotes the sum with the last term halved.

We prove the following theorem.

THEOREM 3.1. *Operator M_n satisfies*

$$
\|I_{4n+1}\|/\sqrt{2} - \delta_n \leqslant \|M_n\| \leqslant (\|I_{4n+1}\|/\sqrt{2} + \|I_{2n}\|)/2, \tag{3.3}
$$

where

$$
\begin{aligned}
\delta_n &= \frac{1}{3\sqrt{2}} \sec\frac{\pi}{12} && \text{for } n = 1, \\
&= \frac{1}{2n+1} + \frac{\sqrt{2}}{2n+1} \left[\frac{1}{2} \sec\frac{\pi}{8n+4} + \sum_{k=1}^{(n-1)/2}{}' \sec\frac{4k-1}{8n+4}\pi \right] && \text{for } n = 3, 5, \ldots, \\
&= \frac{\sqrt{2}}{2n+1} \sum_{k=0}^{n/2} \sec\frac{4k-1}{8n+4}\pi, && \text{for } n = 2, 4, \ldots.
\end{aligned}
\tag{3.4}
$$

Proof. We begin with the proof of the left inequality of (3.3). We show that

$$\|I_{4n+1}\|/\sqrt{2}-\delta_n=\Lambda_{M_n}(0) \qquad \text{for } n \text{ even,}$$
$$=\Lambda_{M_n}(\cos\tau_n) \qquad \text{for } n \text{ odd,}$$

where $\tau_n=\pi/2-\pi/(2n+1)$.

If n is even, it follows from (3.2) (the last but one form) that

$$\Lambda_{M_n}(0)=\frac{2}{2n+1}\sum_{j=0}^{n}{}' \cos\theta_j/|\cos\theta_{j2}|$$
$$=\frac{1}{2n+1}\left\{1+2\sum_{j=1}^{n/2}\left[\cos\frac{1}{2}\theta_{2j}/\cos\theta_{2j}\right.\right.$$
$$\left.\left.-\cos\frac{1}{2}(\pi-\theta_{2j-1})/\cos(\pi-\theta_{2j-1})\right]\right\}.$$

Using the identity

$$\frac{\cos\alpha}{\cos 2\alpha}=\frac{1}{2\sqrt{2}}\left[\csc\left(\frac{\pi}{4}-\alpha\right)+\csc\left(\frac{\pi}{4}+\alpha\right)\right],$$

we obtain

$$\Lambda_{M_n}(0)=\frac{1}{2n+1}\left\{1+\frac{1}{\sqrt{2}}\sum_{j=1}^{n/2}\left[\csc\left(\frac{\pi}{4}-\frac{1}{2}\theta_{2j}\right)+\csc\left(\frac{\pi}{4}+\frac{1}{2}\theta_{2j}\right)\right.\right.$$
$$+\csc\left(\frac{\pi}{4}-\frac{1}{2}\theta_{2j-1}\right)$$
$$\left.\left.-\csc\left(\frac{\pi}{4}+\frac{1}{2}\theta_{2j-1}\right)\right]\right\}$$
$$=\frac{1}{\sqrt{2}(2n+1)}\left\{\sum_{j=1}^{2n+1}\csc\frac{2j-1}{4(2n+1)}\pi-2\sum_{j=1}^{n/2}\csc\frac{2n+4j-1}{4(2n+1)}\pi\right\}$$
$$=\frac{1}{\sqrt{2}}\|I_{4n+1}\|-\frac{\sqrt{2}}{2n+1}\sum_{j=1}^{n/2}\sec\frac{4j-1}{4(2n+1)}\pi.$$

If n is odd then using the last but one expression of (3.2) for the Lebesgue function Λ_{M_n} we obtain

$$\Lambda_{M_n}(\cos\tau_n)=\frac{2}{2n+1}\left\{\left|\sin(2n-1)\frac{\pi}{4}\right|\left(\sum_{k=1}^{(n-1)/2}{}'-\sum_{k=(n+1)/2}^{n}\right)\right.$$
$$\times\frac{\cos\theta_k\sin(1/2)\tau_n}{\cos 2\theta_k-\cos\tau_n}$$

$$- \left|\cos(2n-1)\frac{\pi}{4}\right| \left(\sum_{k=0}^{(n+1)/2}{}' - \sum_{k=(n+3)/2}^{n}\right) \times \frac{\cos\theta_k \cos(1/2)\tau_n}{\cos 2\theta_k + \cos\tau_n}\Bigg\}.$$

Making use of the identity

$$\frac{4\sin\alpha\cos\beta}{\cos 2\beta - \cos 2\alpha} = \csc(\alpha-\beta) + \csc(\alpha+\beta),$$

we have

$$\Lambda_{M_n}(\cos\tau_n) = \frac{1}{2\sqrt{2}(2n+1)}\left\{\left(\sum_{k=0}^{(n-1)/2}{}' - \sum_{k=(n+1)/2}^{n}\right) \times \left[\csc\left(\frac{1}{2}\tau_n - \theta_k\right) + \csc\left(\frac{1}{2}\tau_n + \theta_k\right)\right] + \left(\sum_{k=0}^{(n+1)/2}{}' - \sum_{k=(n+3)/2}^{n}\right)\left[\csc\left(\frac{\pi}{2} - \frac{1}{2}\tau_n - \theta_k\right) + \csc\left(\frac{\pi}{2} - \frac{1}{2}\tau_n + \theta_k\right)\right]\right\}.$$

For $n=1$ the above formula implies

$$\Lambda_{M_1}(\cos\tau_1) = \frac{1}{6\sqrt{2}}\left\{2\sum_{k=1}^{3}\csc\frac{2k-1}{12}\pi - 2\csc\frac{5\pi}{12}\right\} = \frac{1}{\sqrt{2}}\|I_1\| - \frac{1}{3\sqrt{2}}\sec\frac{\pi}{12}.$$

For $n>1$ we have

$$\Lambda_{M_n}(\cos\tau_n) = \frac{1}{2\sqrt{2}(2n+1)}\left\{2\sum_{k=1}^{2n+1}\csc\frac{2k-1}{4(2n+1)}\pi - 4\sum_{k=0}^{(n-1)/2}{}''\csc\frac{2n+4k+1}{4(2n+1)}\pi - 2\csc\frac{4n+1}{4(2n+1)}\pi\right\}$$
$$= \frac{1}{\sqrt{2}(4n+2)}\sum_{k=1}^{4n+2}\csc\frac{2k-1}{2(4n+2)}\pi - \frac{1}{2n+1} - \frac{\sqrt{2}}{2n+1}\left[\sum_{k=1}^{(n-1)/2}{}'\sec\frac{4k-1}{8n+4}\pi + \frac{1}{2}\sec\frac{\pi}{8n+4}\right]$$
$$= \frac{1}{\sqrt{2}}\|I_{4n+1}\| - \delta_n.$$

It remains to prove the right inequality of (3.3). From the last form of the Lebesgue function Λ_{M_n} given in (3.2) it follows that

$$\Lambda_{M_n}(\cos t) \leqslant \frac{1}{4(2n+1)} \phi(t) \qquad (0 \leqslant t \leqslant \pi),$$

where

$$\phi(t) = \left|\sin\left(n+\frac{1}{2}\right)t\right| \sum_{k=-2n-1}^{2n} \left|\tan\frac{1}{4}\left(t+\frac{2k\pi}{2n+1}\right)\right| + \left|\cos\left(n+\frac{1}{2}\right)t\right| \sum_{k=-2n-1}^{2n} \left|\tan\frac{1}{4}\left(t+\frac{2k+1}{2n+1}\pi\right)\right|.$$

It is easily verified that ϕ is a periodic function with the period $\pi/(2n+1)$. Thus we have

$$\|\Lambda_{M_n}\|_\infty \leqslant \frac{1}{4(2n+1)} \max_{0 \leqslant t \leqslant \pi/(2n+1)} \phi(t). \tag{3.5}$$

For $t \in [0, \pi/(2n+1)]$ the function $\phi(t)$ takes the form

$$\begin{aligned}
\phi(t) &= \sin\left(n+\frac{1}{2}\right)t \left\{\tan\frac{1}{4}t + \sum_{k=1}^{2n}\left[\tan\frac{1}{4}\left(\frac{2k\pi}{2n+1}-t\right)+\tan\frac{1}{4}\left(\frac{2k\pi}{2n+1}+t\right)\right] \right.\\
&\quad \left. +\tan\frac{1}{4}(2\pi-t)\right\} + \cos\left(n+\frac{1}{2}\right)t \\
&\quad \times \sum_{k=1}^{2n+1}\left[\tan\frac{1}{4}\left(\frac{2k-1}{2n+1}\pi-t\right)+\tan\frac{1}{4}\left(\frac{2k-1}{2n+1}\pi+t\right)\right] \\
&= \sin(2n+1)\frac{t}{2}\sum_{k=1}^{2n+1}\left[\tan\frac{1}{2}\left(\frac{k\pi}{2n+1}-\frac{1}{2}t\right)\right. \\
&\quad \left. +\tan\frac{1}{2}\left(\frac{k-1}{2n+1}\pi+\frac{1}{2}t\right)\right] \\
&\quad + \sin\left[(2n+1)\left(\frac{\pi}{2(2n+1)}-\frac{1}{2}\right)t\right] \\
&\quad \times \sum_{k=1}^{2n+1}\left[\tan\frac{1}{2}\left(\frac{k-1}{2n+1}\pi+\frac{1}{2}\left[\frac{\pi}{2n+1}-t\right]\right)\right. \\
&\quad \left. +\tan\frac{1}{2}\left(\frac{k\pi}{2n+1}-\frac{1}{2}\left[\frac{\pi}{2n+1}-t\right]\right)\right] \\
&= \psi\left(\frac{1}{2}t\right)+\psi\left(\frac{1}{2}\left[\frac{\pi}{2n+1}-t\right]\right),
\end{aligned}$$

where

$$\psi(u)=\sin(2n+1)\,u\sum_{k=1}^{2n+1}\left[\tan\frac{1}{2}\left(\frac{k\pi}{2n+1}-u\right)+\tan\frac{1}{2}\left(\frac{k-1}{2n+1}\pi+u\right)\right]$$

$$\left(0\leqslant u\leqslant\frac{\pi}{2(2n+1)}\right).$$

McCabe and Phillips [7] have shown that ψ is monotonically increasing on $[0, \pi/2(2n+1)]$. Thus

$$\max_{0\leqslant t\leqslant\pi/(2n+1)}\phi(t)\leqslant\psi\left(\frac{\pi}{4(2n+1)}\right)+\psi\left(\frac{\pi}{2(2n+1)}\right)$$

$$=\frac{1}{\sqrt{2}}\sum_{k=1}^{4n+2}\tan\frac{2k-1}{8(2n+1)}\pi+2\sum_{k=1}^{2n+1}\tan\frac{2k-1}{4(2n+1)}\pi$$

$$=2(2n+1)\{\|I_{4n+1}\|/\sqrt{2}+\|I_{2n}\|\}.$$

Hence, in view of (3.5), follows the right inequality of (3.3). ∎

Results of numerical calculations, reported in Table III, allow us to conjecture that

$$\|M_n\|=\Lambda_{M_n}(1)=\tfrac{1}{2}(1+\|I_{2n}\|)\qquad(n=1,2,\ldots,7),$$
$$=\|I_{4n+1}\|/\sqrt{2}-\delta_n\qquad(n=8,9,\ldots),$$

δ_n being given by (3.4).

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